Preview

To the Infinite and Back Again

A Workbook in Projective Geometry

Part I

HENRIKE HOLDREGE

EVOLVING SCIENCE ASSOCIATION

A collaboration of The Nature Institute
and The Myrin Institute

2019

Table of Contents

Chapter 1

THE HARMONIC NET AND THE HARMONIC FOUR POINTS

The elements projective geometry works with include points at infinity, lines at infinity, and the plane at infinity. In Euclidean geometry (after Euclid, Greek mathematician, 300 BCE), which is taught in middle and high schools, these concepts are not known. Euclidian geometry states, for instance, that two lines in a plane have a point in common unless they are parallel. For projective geometry, any two lines in a plane have a point in common, and for parallel lines this is a point at infinity.

That parallel lines should have a point in common is a strange and challenging idea, and for us, at first, it is not supported by any experience. In this chapter, I will attempt to provide a geometrical context and a thought experience by which to approach this idea. For this context I have chosen the harmonic net and the law of the harmonic four points.

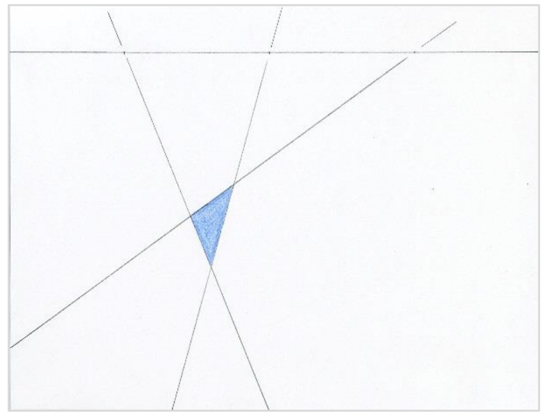
I will give instructions for geometric drawings, and I strongly recommend that you, the reader, actually do the drawings. The figures and photos given in the text are meant as an aid for clarification and cannot be a substitute for your own work and engagement.

You will need paper (letter-size paper will be sufficient for most drawings), a compass, a straight edge (ruler), a well-sharpened pencil and a few colored pencils, eraser, pencil sharpener, and maybe a board to place the paper on. Most of all, you will need quiet time. I hope you enjoy the exercises and the discoveries you will make.

Course on projective geometry at The Nature Institute *in spring 2015*

Lesson 1: Drawing a harmonic net

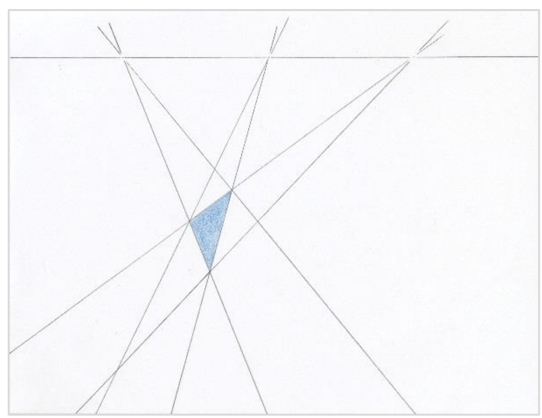
With your paper placed horizontally, draw a line from edge to edge. The line should be parallel to the upper edge and about an inch away from it. Choose three points on this horizontal line, so that the point in the middle is equidistant from the other two points.



[To achieve this you can use the compass: Mark a point somewhere near the middle of the line. Open your compass and place the compass needle on the point you just marked. With the compass pencil, mark the points to the left and the right without changing the compass opening.]

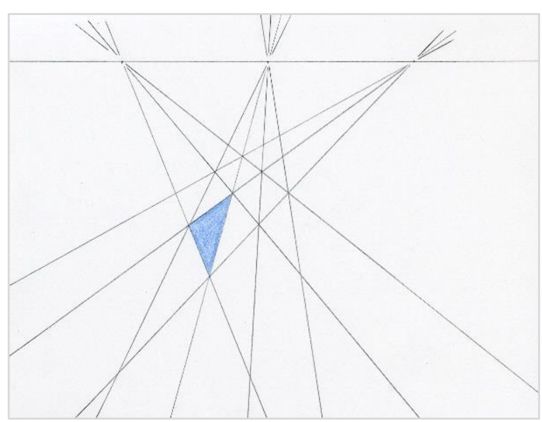
Next, draw a line through each of the three points such that the three lines form a triangle below the horizontal line. Draw these and all following lines all the way to the edge of the paper. With a colored pencil, shade the triangle. (Your triangle need not resemble the one I have drawn.)

At this point you will have made all the free choices there are. From now on, every line that you will draw will be predetermined. As your drawing unfolds, step by step, line by line, you will make visible what in a sense is already there, albeit invisible.



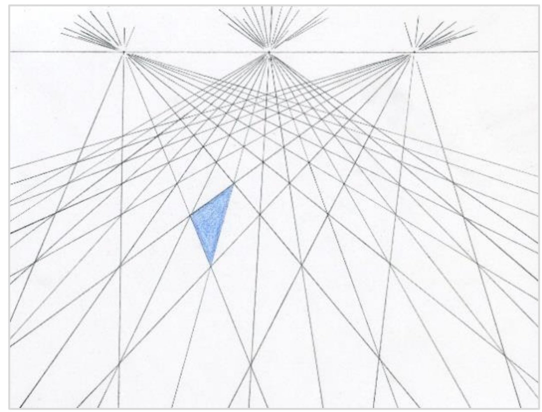
To draw the next lines, take a look at your shaded triangle. Each of its corners is connected with two of the three points on the horizontal line, but not connected with one of the three. Draw for each corner the line that connects it with the point on the horizontal line it is not yet connected with.

You have now added three more lines to your drawing. They create new points of intersection which, again, are not connected with one of the three points on the horizontal line. Draw for every new intersection the third line.



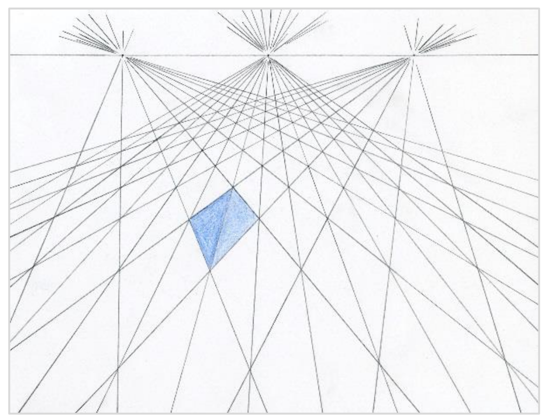
Continue the process in this way, radiating out from the shaded triangle. Draw line after line. Proceed methodically; do not overlook a step.

You will soon find that several points are on the same line. However, they might seem to be not exactly on one line. Here is where you might misjudge the situation and make a mistake. It would be a mistake if you now drew several lines, more or less close to each other. Instead, you have to draw just one line with the points being more or less well met. It is our inability to draw with greater accuracy that causes this problem. It is not inherent in the construction itself. You will find these inaccuracies more pronounced toward the bottom and the sides of the paper, and less toward the horizontal line.



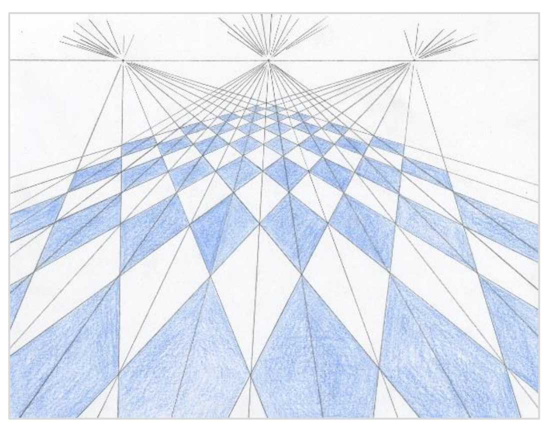
Continue drawing and enjoy the lawful unfolding of the net of lines. Toward the bottom and the sides of the paper you can complete the work; but upwards, toward the horizontal line, the process is never ending. You have to decide when to stop.

Looking at the drawing, you see that from each of the three points on the horizontal line there issues a set of lines. Together they form a pattern of triangles of which the initial shaded triangle is one. There are no gaps within this pattern, and each triangle is bordered by three other triangles.



We will now organize the drawing into a net of quadrilaterals. A quadrilateral is a four-sided figure, bounded by four lines. A square, a rectangle and a parallelogram, for instance, are quadrilaterals. In projective geometry, where sizes of angle openings and lengths of line segments are of no concern, we work with quadrilaterals of any shape and form. Every quadrilateral has two pairs of opposite sides, and every quadrilateral has two diagonals that connect opposite corners.

The initial shaded triangle forms, together with any of its three neighboring triangles, a quadrilateral. Find the neighboring triangle that shares with the initial shaded one the line that passes through the mid-point on the horizontal line. Shade the entire quadrilateral.

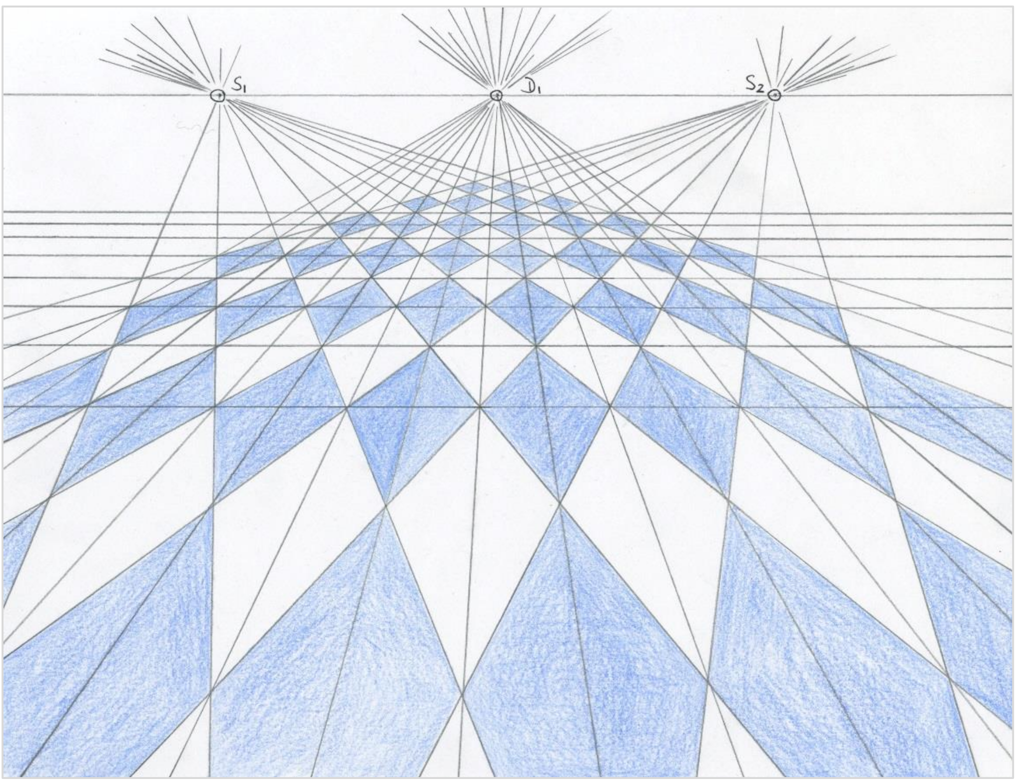


Through shading, organize the net of triangles into a net of quadrilaterals as far as the construction has been developed. Each immediately neighboring quadrilateral stays blank, the next one will be shaded. The result will resemble a checker board seen in perspective with shaded and non-shaded parts alternating with each other.

We now see that the three points on the horizontal line have different relations to the quadrilaterals. Two of them, the points to the left and the right, each produce lines forming opposite sides of the quadrilaterals. I designate these points as side-points S_1 and S_2. The midpoint issues diagonals, and I designate it as diagonal-point D_1.

For each quadrilateral, only one of its two diagonals is drawn as part of the construction. The other one is missing. As a final step, draw these diagonals. (For accuracy, utilize the points in your drawing that are most precise.)

Sufficient precision provided, you will find that the lines you now added are all parallel to each other and parallel to the initial horizontal line.



When working in a group, it is worthwhile to look at all the completed drawings and compare them. All drawings are constructed following the same principle. However, they might look markedly different. Depending on the choice of the initial triangle in its size, shape, and position, the drawings differ in perspective—they appear as if a tiled floor was looked at from a higher or a lower vantage point.

When you cover up the areas in your drawing where the quadrilaterals are most distorted, you will see the resemblance with tiled floors in paintings, for instance, by the Dutch masters Johannes Vermeer and Pieter de Hooch (see page 10). The horizontal line in the construction of a harmonic net is, in perspectival representations, the horizon line, and the three points on it are vanishing points. You will meet vanishing points and the horizon line again in chapter 6.